

# The Use of Decomposition SVD to Approximate a Surface

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**Abstract.** This paper describes the procedure of determining the parameters of the approximating function of the surface using a distribution of special value. From a practical point of view, an important issue is to determine the covariance matrix of the estimated parameters. Interval estimation was carried out and a methodology to obtain an optimal equation of approximating surface was presented. The main emphasis was given to the elimination of these parameters functions that generate unnecessary disturbance. The results obtained using the approximate decomposition SVD were compared with those obtained by a classical method of least squares. They have been used a variety of software, including software written by author, also packages Matlab and Statistica. The main purpose of discussion is to solve sample tasks for better understanding and expand the use of decomposition SVD in geodetic issues.

**Keywords:** Singular value decomposition, approximation, the equation surface, interval estimation, geodesy.

**Conference topic:** Technologies of Geodesy and Cadastre.

## Introduction

Approximation is a representation of the function  $y = f(x)$  using simpler, belonging to a certain class of functions  $y = F(x)$ . In surveying issues often known only finite (discrete) set of values of the function and purpose of approximation is forecasting value, including the assessment of the accuracy of the forecast (Hanus *et al.* 2014; Jasińska 2012). Function approximating (zooming) is looking mostly in a family functions, for example, among polynomials and exponential functions (Korn, G. A., Korn, T. M. 1983). Bringing one function by another always results in the appearance of inaccuracies approximation. Depending on the method of measuring inaccuracy of the approximation distinguishes:

– Uniform approximation – it is assumed that the functions  $y = f(x)$  and  $y = F(x)$  are defined and continuous in the range of  $[a; b]$ . The accuracy of approximation is measured by Chebyshev’s standard:

$$\|f - F\| = \sup_{a \leq x \leq b} \|f(x) - F(x)\|;$$

– Medium-square continuous approximation-function  $y = f(x)$  is defined and continuous in the range of  $[a; b]$ . The accuracy of approximation is measured by:

$$\|f - F\| = \int_a^b p(x) [f(x) - F(x)]^2 dx,$$

where  $p(x)$  – non-negative, the actual weighting function.

– Medium-square discrete approximation – known function values can be represented by the table:

$x_i$	$x_1$	$x_2$	...	$x_n$
$y_i = f(x_i)$	$y_1$	$y_2$	...	$y_n$

The accuracy of approximation is measured by:

$$\|f - F\| = \sum_{i=1}^n p(x_i) [f(x_i) - F(x_i)]^2.$$

The function approximating is selected most often in the form of generalized polynomial  $F(x) = a_0\varphi_0(x) + a_1\varphi_1(x) + \dots + a_m\varphi_m(x)$ , in which functions  $\varphi_0(x), \varphi_1(x), \dots, \varphi_m(x)$  are selected basis ‘a priori’

functions,  $m + 1$  dimensional linear space. In this case, the task of approximation is reduced to the determination of the  $a_0, a_1, \dots, a_m$ . As a basis function, most often monomial or trigonometric functions are chosen. According to Weierstrass claims, for every function  $y = f(x)$  defined and continuous on the closed and bounded stretch  $[a; b]$  there is a polynomial  $W_m = a_0 + a_1x + a_2x^2 + \dots + a_mx^m$ , that brings uniformly function  $y = f(x)$  section  $[a; b]$ , and for every function  $y = f(x)$  defined and continuous on  $R$  and periodic of period  $2\pi$ , there is a polynomial trigonometric  $S_m(x) = \frac{a_0}{2} + \sum_{k=1}^m a_k \cos(kx) + b_k \sin(kx)$ , which brings uniformly function  $y = f(x)$ .

In the case where approximation is performed to estimate (e.g., displacement values at the time), then it plays an important role reliability of forecast. The accuracy of predictions determined by the rules of propagation of covariance is usually highest in the case of the simplest functions approximating, unfortunately not always the reliability of such forecasts can be considered sufficient (Preweda 2013). The essence undertaken subject is the application of discrete regularization to minimize the impact of noise occurring in the observations. Thanks to this operation, approximating functions can be made simpler to increase the credibility of forecasts. Application of SVD decomposition (Gloub, Reinsch 1970) is known primarily from solving poorly conditioned systems of equations or, for example, image compression. In this work demonstrates the possibility of using this distribution for the prediction based on the “noisy” data input. Of course, there are other mitigating “noise”, such as estimation methods strong. In this case, we focus exclusively on the regularization of the task in terms of prediction, although the numerical point of view, these tasks can even be well conditioned.

### Discrete regularization task of approximation using SVD

Approximation of curves and surfaces is reduced to solution of the problem of least squares, which can be stored in the known matrix form:

$$\mathbf{Ax} = \mathbf{L} \leftarrow \mathbf{P}, \tag{1}$$

where  $\mathbf{A}$  – the coefficients matrix;  $\mathbf{L}$  – the free words vector;  $\mathbf{P}$  – matrix of weights;  $\mathbf{x}$  – vector of unknowns.

According to the method of least squares unknowns estimator  $\hat{\mathbf{x}}$  is determined from the relationship

$$\hat{\mathbf{x}} = (\mathbf{A}^T \mathbf{PA})^{-1} \mathbf{A}^T \mathbf{PL}. \tag{2}$$

Let there be a function  $f(x_i, y_i) = dz_i$  (see Table 1).

Table 1. Function  $f(x_i, y_i) = dz_i$

$x_i$	0.7	0.0	0.6	0.1	10.0	10.7	10.5	10.1	20.7	20.6	20.7	20.7	30.3	30.8
$y_i$	0.0	10.5	20.0	30.2	0.0	10.5	20.2	30.2	0.1	10.2	20.7	30.1	0.1	10.3
$dz_i$	-21.9	-12.2	-27.0	-31.3	-17.2	-17.6	-30.8	-43.2	-1.4	-15.9	-36.6	-74.3	-33.5	-35.2
$x_i$	30.0	30.1	40.6	40.7	40.8	40.5	50.7	50.1	50.2	50.2	50.3	50.2	50.7	50.7
$y_i$	20.0	30.1	0.3	10.5	20.7	30.2	0.2	10.3	20.4	30.3	0.6	10.6	20.6	30.2
$dz_i$	-52.0	-64.0	0.5	-19.6	-61.7	-84.5	-9.2	-19.3	-70.1	-82.3	14.7	-33.7	-66.2	-98.9

For simplicity, suppose that the observations  $y_i$  have equal accuracy ( $\mathbf{P} = \mathbf{I}$ ). We assume that we approximate quadratic surface:

$$dz_i = a_0 + a_1x_i + a_2y_i + a_3x_i^2 + a_4y_i^2 + a_5x_iy_i \quad \text{for } i = 1 \dots 28. \tag{3}$$

The solution of the above system is burdened inaccuracy arising due to inaccuracy of observations and numerical calculations. In terms of forecast, it will also be subject to uncertainty arising from the adoption of a specific function approximating which is determined because of the preliminary analysis, but in practice it is never recognized until the end. Input errors have of importance in the approximate area of the second degree (Preweda 1993; Jasińska *et al.* 2003; Jasińska, Preweda 2004, 2012).

In other cases, relative small data deficits substantially affect estimated uncertainties vector, resulting in relatively large relative changes of the solution. This sensitivity sought solutions to changes in the parameters is referred to as a conditioning assignment. For the evaluation of the effect of disturbance data disorders solution using various numerical values hereinafter conditions indicators (Wilkinson 1965; Lawson, Hanson 1965) Based on years of the author's experience it may be said that the use of not one, but several basic indicators allow proper assessment of conditions of the task and take the appropriate decision on how to solve it. Another problem is the issue of the forecast based on the function approximating. In this case, too, you can be successfully applied regularization discreate, even despite the good conditions of the task, particularly when the forecast must be based on substantial extrapolation. In the analysis of the problem in terms of numerical indicators are helpful Turing's  $M(\mathbf{Z})$  and  $N(\mathbf{Z})$  and Von Neuman'a-Goldstin's  $P(\mathbf{Z})$ . These ratios are determined from the relationship:

$$\begin{aligned} M(\mathbf{Z}) &= n \cdot \max_{i,j} |a_{ij}| \cdot \max_{i,j} |\alpha_{ij}|; \\ N(\mathbf{Z}) &= n^{-1} \cdot \|\mathbf{Z}\| \cdot \|\mathbf{Z}^{-1}\|; \\ P(\mathbf{Z}) &= \left| \frac{\lambda_{\max}}{\lambda_{\min}} \right|, \end{aligned} \tag{4}$$

where:  $\mathbf{Z} = (a_{ij})$  – the square matrix  $n \times n$ ;  $\mathbf{Z}^{-1} = (\alpha_{ij})$  – the inverse matrix  $\mathbf{Z}$ ;  $\|\mathbf{Z}\| = \left( \sum_{ij} a_{ij}^2 \right)^{\frac{1}{2}}$ ,

$\|\mathbf{Z}^{-1}\| = \left( \sum_{ij} \alpha_{ij}^2 \right)^{\frac{1}{2}}$ ,  $\lambda_{\max}, \lambda_{\min}$  – the largest and the smallest eigenvalue of the matrix  $\mathbf{Z}$ .

For a well-conditioned matrix, appropriate indicators should be of the order:

$$M(\mathbf{Z})_{\text{opt}} = \sqrt{n} \log n, \quad N(\mathbf{Z})_{\text{opt}} = \sqrt{n}, \quad P(\mathbf{Z})_{\text{opt}} = n.$$

Obtaining such indicators of conditions in practice almost impossible, and the possibility of comparison with “standard” is very valuable to take appropriate action in the search for optimal estimators. From a practical point of view, it should be noted that between indicators of conditions occur relationships:

$$\frac{M(\mathbf{Z})}{n^2} \leq N(\mathbf{Z}) \leq M(\mathbf{Z}) \quad (\text{for } n \geq 10);$$

$$\frac{M(\mathbf{Z})}{n} \leq P(\mathbf{Z}) \leq nM(\mathbf{Z}).$$

Let's check conditioning matrix for the matrix obtained based on (3):

$$\mathbf{Z} = \begin{bmatrix} 28,0000 & 812,3000 & 428,1000 & 32731,6100 & 10043,6500 & 12450,8800 \\ 812,3000 & 32731,6100 & 12450,8800 & 1443266,2910 & 291934,6480 & 502615,5640 \\ 428,1000 & 12450,8800 & 10043,6500 & 502615,5640 & 259658,5410 & 291934,6480 \\ 32731,6100 & 1443266,2910 & 502615,5640 & 66648286,2353 & 11788328,4680 & 22185574,3648 \\ 10043,6500 & 291934,6480 & 259658,5410 & 11788328,4680 & 7101264,4861 & 7550545,1248 \\ 12450,8800 & 502615,5640 & 291934,6480 & 22185574,3648 & 7550545,1248 & 11788328,4680 \end{bmatrix};$$

$$M(\mathbf{Z}) = 194213628;$$

$$N(\mathbf{Z}) = 6306873;$$

$$P(\mathbf{Z}) = 38746198.$$

Optimal values are:  $M(\mathbf{Z})_{\text{opt}} = 4.39$ ,  $N(\mathbf{Z})_{\text{opt}} = 2.45$ ,  $P(\mathbf{Z})_{\text{opt}} = 6$ . The matrix  $\mathbf{A}^T \mathbf{A}$  is not well conditioned in terms designate the reverse, however, you can solve the task using a least squares method:

$$\hat{x} = \begin{matrix} & P = 0.68 & P = 0.90 \\ \begin{bmatrix} -16.51701 & \pm 6.902 & \pm 11.851 \\ 0.00040 & \pm 0.435 & \pm 0.747 \\ 0.17351 & \pm 0.645 & \pm 1.107 \\ 0.00698 & \pm 0.007 & \pm 0.013 \\ -0.02096 & \pm 0.019 & \pm 0.032 \\ -0.05417 & \pm 0.009 & \pm 0.016 \end{bmatrix} \end{matrix}$$

The standard deviations of the model parameters determined at a probability level of P = 0.68 and a probability level of P = 0.90 (based on interval estimation based on the distribution t- Student). Quadratic surface model is shown in Figure 1.

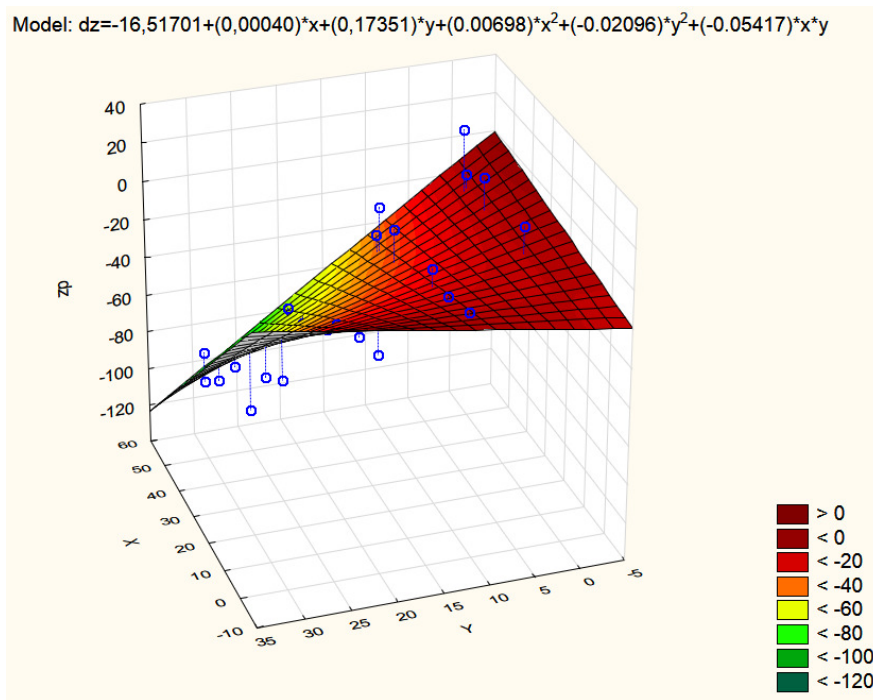


Fig. 1. Approximation quadratic surface

Verification of statistical hypotheses about the importance of the parameters shows that the probability level P = 0.68, statistically significant only three parameters ( $a_0, a_4, a_5$ ). Based on approximation of the model surface:

$$dz_i = a_0 + a_4 y_i^2 + a_5 x_i y_i \quad \text{for } i = 1 \dots 28, \quad (5)$$

the following results were obtained:

$$\hat{x} = \begin{matrix} & P = 0.68 & P = 0.90 \\ \begin{bmatrix} -9.59372 & \pm 2.895 & \pm 4.945 \\ -0.02918 & \pm 0.007 & \pm 0.013 \\ -0.03877 & \pm 0.006 & \pm 0.009 \end{bmatrix} \end{matrix}$$

All parameters of the model are statistically significant. Let's see yet approximation plane of model:

$$dz_i = a_0 + a_1 x_i + a_2 y_i \quad \text{for } i = 1 \dots 28; \quad (6)$$

$$\hat{x} = \begin{matrix} & P = 0.68 & P = 0.90 \\ \begin{bmatrix} 6.83031 & \pm 6.678 & \pm 11.407 \\ -0.45317 & \pm 0.159 & \pm 0.271 \\ -2.02649 & \pm 0.257 & \pm 0.438 \end{bmatrix} \end{matrix}$$

Approximation of the model shown in Figure 2.

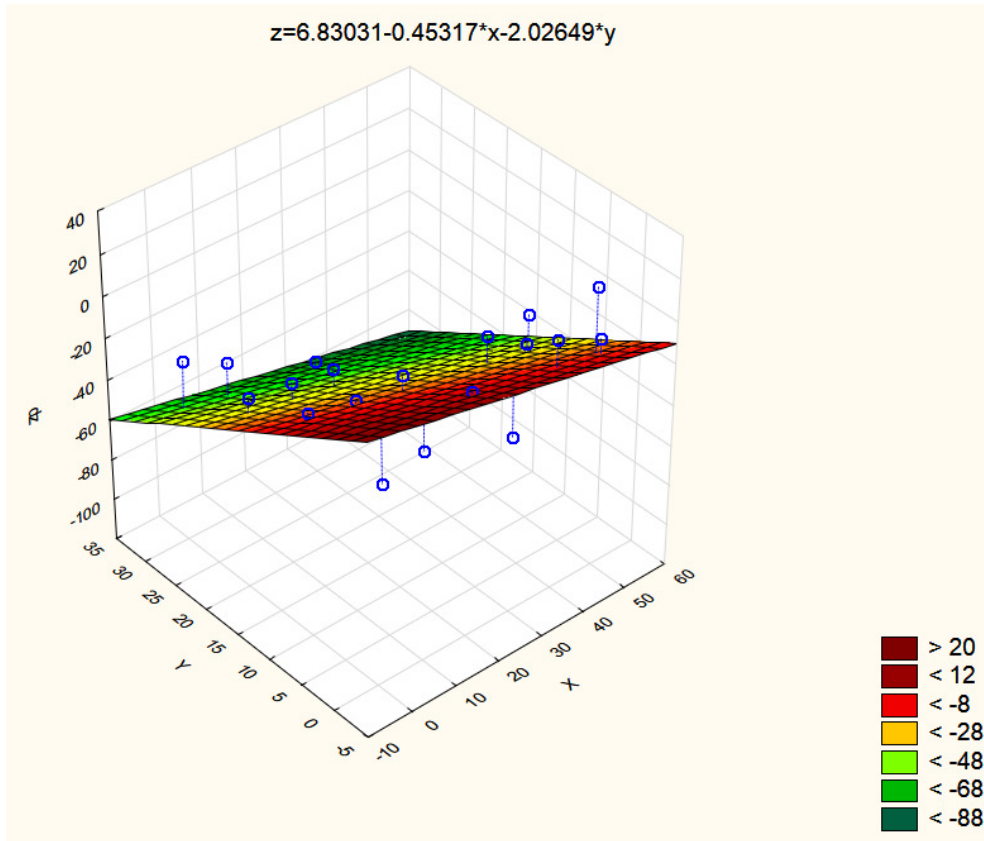


Fig. 2. Approximation of surface per equation:  $dz = a_0 + a_1x + a_2y$

The purpose of discussion is a regularization of the task. The basis of regularization will be a discrete distribution coefficient matrix approximation equations singular value. Because of this distribution, we receive practically always a positive singular value the angle of the value may be near zero. The number of these almost zero singular value depends on the type, the accuracy of measurement and accuracy of calculation. Some parameters of the model can be very sensitive to the type of disorder input. The task can be solved correctly provided knowledge of the actual order of the matrix coefficients. Suppose that the matrix  $A$  is not a regular matrix. If the rank of this matrix is known a priori and should be  $p$ , then the correct solution simply equate  $(n-p)$ , the smallest singular value to zero. Matrix  $S = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$  rank  $r(S) = u$  we replace by matrix  $S_\alpha$  which is:  $S_\alpha = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_p, 0, \dots, 0)$ . Rank of this matrix  $p$  called pseudo-rank of the matrix  $A$ . Number  $\alpha$  satisfying the relationship  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_p \geq \alpha \geq \sigma_{p+1} \geq \dots \geq \sigma_n$  referred to as the parameter of regularization.

Pseudo-inverse of regularized matrix  $A_\alpha = US_\alpha V^T = A + dA_\alpha$  is expressed by the formula  $A_\alpha^+ = VS_\alpha^+ U^T$ , Estimated uncertainties while the vector is determined from the relationship:

$$\hat{x}_\alpha = A_\alpha^+ L. \quad (7)$$

Let us determine the singular values of the matrix  $A$  for the Eq. (3):

$$S = \text{diag}(77191297.53, 7333923.04, 1053817.21, 1320.77, 321.85, 2.05)$$

Based on the value analysis  $|\sigma_{\hat{x}_i} / \hat{x}_i|$  assume pseudo rank of the matrix  $p(A) = 5$ .

Hence  $S_\alpha = \text{diag}(77191297.53, 7333923.04, 1053817.21, 1320.77, 321.85, 0)$ , while

$$\hat{\mathbf{x}}_{\alpha} = \mathbf{A}_{\alpha}^{+} \mathbf{L} = \begin{bmatrix} -0.06886 \\ -0.70488 \\ -0.71543 \\ 0.01443 \\ -0.00883 \\ -0.04125 \end{bmatrix}$$

It should be noted that the thus obtained unknowns do not meet the condition of the least squares method, due to regularization of the task. It can be shown that the sum of deviations calculated as  $\hat{\mathbf{v}}_{\alpha} = \mathbf{A}\hat{\mathbf{x}}_{\alpha} - \mathbf{L}$  will be different from zero. The final solution must be corrected by a factor of systematic resulting from regularization. To eliminate this factor, in this task simply subtracted from the estimated parameter  $a_0 = \hat{x}_{\alpha_1}$  the average value calculated from the vector  $\hat{\mathbf{v}}_{\alpha}$ . Model parameters after eliminating systematic factor resulting from regularization tasks are:

$$\hat{\mathbf{x}} = \begin{matrix} P = 0.68 & P = 0.90 \\ \begin{bmatrix} -1.27268 & \pm 0.028 & \pm 0.047 \\ -0.70488 & \pm 0.019 & \pm 0.033 \\ -0.71543 & \pm 0.118 & \pm 0.202 \\ 0.01443 & \pm 0.007 & \pm 0.013 \\ -0.00883 & \pm 0.020 & \pm 0.034 \\ -0.04125 & \pm 0.008 & \pm 0.014 \end{bmatrix} \end{matrix}$$

This model is illustrated in Figure 3.

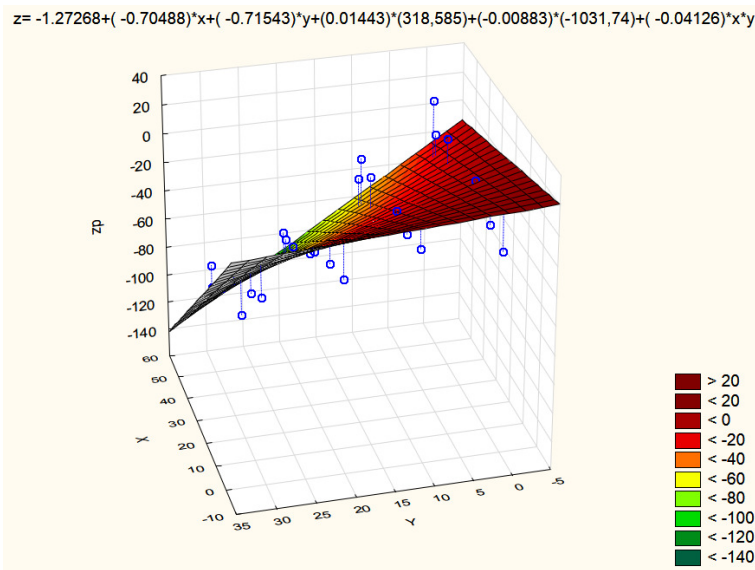


Fig. 3. Approximation of the area quadratic after regularization

Table 2 summarizes the results of four prediction models.

Table. 2. The forecast by 4 models of approximation (x = 15, y = 15)

Model	R <sup>2</sup>	dz [mm]	mz [mm]
$dz_i = a_0 + a_1x_i + a_2y_i + a_3x_i^2 + a_4y_i^2 + a_5x_iy_i$	0.96	-29.2	3.6
$dz_i = a_0 + a_4y_i^2 + a_5x_iy_i$	0.94	-24.9	2.2
$dz_i = a_0 + a_1x_i + a_2y_i$	0.74	-30.4	3.6
$dz_i = a_0 + a_1x_i + a_2y_i + a_3x_i^2 + a_4y_i^2 + a_5x_iy_i + \text{regularization}$	0.96	-30.6	3.9

## Conclusions

In practice, you can only roughly determine the correct row of the matrix coefficients of the approximation equations. The disorder of satisfies  $|\delta\sigma_i| \leq \|dA\|$ , while the estimate disorder regularized matrix  $A_\alpha$  can be written as:  $\|dA_\alpha\| \leq \varepsilon + \sigma_{p+1} \leq \varepsilon + \alpha$ . Parameter  $\varepsilon = \|dA\|$  determines the level of disturbance matrix  $A$ . Since the pseudo-inverse of the regularized  $A_\alpha^+$  should fulfil the Lipschitz condition of continuity, therefore, the inequality  $\|A^+\| \|dA_\alpha\| < 1$  must be true. It seems that  $\|A\| + \|dA_\alpha\| \leq \frac{\varepsilon + \alpha}{\sigma_r}$ , wherein  $\sigma_r$  is the smallest, different from zero, singular value does not impaired matrix  $A$ . If we choose the regularization parameter, so that was fulfilled relationship  $\alpha \geq \varepsilon$ , then will take place the equality of rows of the regularized matrix  $A_\alpha$  and matrix  $A$ . The upper value of parameter estimation  $\alpha$  can be written as  $\alpha < \sigma_r - \varepsilon$ , from with the result that  $\varepsilon \leq \alpha \leq \sigma_r - \varepsilon$ . With this relationship, can be concluded that a necessary condition for its fulfilment is that the relationship occurred  $2\varepsilon < \sigma_r$ . The paper arbitrarily assumed a rank of coefficient matrix as one less than the actual rank, to demonstrate the possibility of an intermediate solution of the problem, for example, between the polynomial and  $n$  ( $n-1$ ) degree. The solution is based on the regularization of discrete suited much better to predict than based on reducing or increasing the degree compile function approximating. This solution is also indispensable in case of an actual fault matrix of coefficients of approximation equations, resulting from a defect or the defect numeric.

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