M$_{\text{split}}$ and M$_{\mathcal{P}}$ estimation. A wider range of robustness

Robert Duchnowski[1], Zbigniew Wiśniewski[2]

Institute of Geodesy, University of Warmia and Mazury, Olsztyn, Poland
E-mails: [1]robert.duchnowski@uwm.edu.pl (corresponding author); [2]zbyszekw@uwm.edu.pl

Abstract. M$_{\text{split}}$ and M$_{\mathcal{P}}$ estimations are new methods of assessing the parameters of functional models of geodetic observations. The first method assumes that each observation can be assigned to either of some functional models which differ from each other in competitive parameters. While the latter method is based on the assumption that distributions of measurement errors differ from the normal one in asymmetry and excess kurtosis. The theoretical properties indicate that both methods are also robust against outliers. However, the sense of robustness is a little wider than in the case of M-estimation. In M$_{\text{split}}$ estimation the outliers are treated as variables with competitive functional models (in relation to models of "good" observations) while robustness of M$_{\mathcal{P}}$ estimation depends on the mentioned parameters of probabilistic models of observations. This paper shows that on one hand robustness is an interesting property of the methods in question, but on the other hand it broadens possible application of such estimation methods.

Keywords: M$_{\text{split}}$ and M$_{\mathcal{P}}$ estimation, robustness.

Conference topic: Technologies of Geodesy and Cadastre.

Introduction

The influence function and the respective weight function play a very important role in designing robust M-estimates. Huber's or Hample's functions (Huber 1981; Hampel *et al.* 1986) as wella s the Danish' one (Kubik 1982) are the most popular functions which can be applied during adjustment of geodetic observations. M$_{\text{split(q)}}$ estimation is another development of M-estimation and it was proposed by Wiśniewski (2009, 2010). The method's main assumption is that the observation set is a mixture of observations with different location parameters. In the context of robustness, we can say that such a set is a mixture of "good" and outlying observations. Note that here we estimate the parameters of both such subsets. M$_{\text{split(q)}}$ estimation was applied as a robust method in some papers (e. g., Li *et al.* 2013; Janicka, Rapiński 2013; Janowski, Rapiński 2013; Błaszczak-Bąk *et al.* 2015), and in the others it was indicated as an alternative to classical M-estimation (Yang *et al.* 2001; Ge *et al.* 2013; Amiri-Simkooei *et al.* 2016). M$_{\text{split(q)}}$ estimation has also several other applications, for example, in analysis of deformation of geodetic networks (Duchnowski and Wiśniewski 2011, 2014, 2016; Zienkiewicz 2015; Zienkiewicz, Baryła 2015; Wiśniewski, Zienkiewicz 2016).

Another new development of classical M-estimation is M$_{\mathcal{P}}$ estimation, which was proposed by Wiśniewski (2014). The method is based on the maximum likelihood method (ML) when the system of Pearson's distributions is assumed as a general probabilistic model of measurement errors. M$_{\mathcal{P}}$ estimation allows us to consider some anomalies of error distribution, namely asymmetry and excess kurtosis (in relation to the normal distribution). To find such anomalies one can apply, for example, the method proposed by Wiśniewski (1996).

It is worth noting that robustness of both methods in question should rather be regarded as a "side effect", and not the main reason for their development. However, the empirical analyses and applications show that both methods can be an interesting alternative for robust M-estimation, especially if we consider their high robustness against outliers and other properties which are specific for each of the methods.

Qualitative robustness of M-estimators

M-estimators of the parameters $\mathbf{X}$ in the observation model $\mathbf{y} = \boldsymbol{\theta} + \mathbf{v} = \mathbf{AX} + \mathbf{v}$, $\boldsymbol{\theta} = \mathbf{AX}$, are such $\hat{\mathbf{X}}$, for which it holds

$$\min_{\mathbf{X}} \varphi(\mathbf{X}) = \min_{\mathbf{X}} \left(\sum_{i=1}^{n} \rho(y_i, \mathbf{a}_i \mathbf{X}) \right) = \sum_{i=1}^{n} \rho(y_i, \mathbf{a}_i \hat{\mathbf{X}}), \tag{1}$$

where: $\mathbf{y} = [y_1, \cdots, y_n]^T$ – an observation vector, $\mathbf{v} = [v_1, \cdots, v_n]^T$ – a vector of random measurement errors, $\mathbf{X}$ – a parameter vector, $\mathbf{A}$ – a known matrix of coefficients, $\mathbf{a}_i$ – ith row of the matrix $\mathbf{A}$ ($\theta_i = \mathbf{a}_i \mathbf{X}$). According to the

classical M-estimation, the functions $\rho(y_i, \theta_i) = \rho(v_i)$ are related to the ε– contaminated model $F_\varepsilon = (1-\varepsilon)F + \varepsilon G$, where $F = F(y_i; \theta_i)$ is an acceptable distribution, namely distribution of good observations, and G are disturbing distributions (distributions of outliers). The coefficient $\varepsilon \in \langle 0; 1 \rangle$ determines the share of F or G within the model F_ε. Its interpretation is asymptotic breakdown point ε^*, which determines quantitative robustness of the estimate $\hat{\theta} = \hat{\theta}(F_\varepsilon)$, namely the boundary share of outliers within the observation set for which the estimate does not break down. On the other hand qualitative robustness is related to sensitivity to gross errors. If the functions ρ are differentiable, then the influence functions $\psi = \rho'$ as well as the weight functions $w = \psi / v$ can be determined (Hampel *et al.* 1986). The influence function is the basis for determination of the maximum sensitivity to gross error (e.g. Huber 1981)

$$\gamma^* = \max\{\gamma_+^*, -\gamma_-^*\}, \tag{2}$$

where $\gamma_-^* = \inf \psi(v)$ and $\gamma_+^* = \sup \psi(v)$ are right hand or left hand sensitivity, respectively. If the influence functions are bounded and the weight ones are concave, then M-estimate is robust against outliers. But if $\gamma^* = \infty$ and the weight function is convex then the estimate is weak. It means that it strongly depends on the outlying observations. In the case of the least squares method (LS), the influence function is unbounded but the weight function is constant for all $v \in (-\infty, \infty)$. In such a case, M-estimates (including LSE) are neutral on outliers. **M_{split} estimation and its robustness**

The basis for M_{split} estimation is an assumption that the observation set is a mixture of several random variables which differ from each other in the parameters $\theta_{i(l)}$, $l = 1, \ldots, q$ (Wiśniewski 2009, 2010). However, we do not know which parameter is proper for a particular observation y_i. Let us assume that we have good observations with the parameter $\theta_{(1)}$ and outlying observations with the parameter $\theta_{(2)}$. Thus each observation can be a good one or outlying one. Hence, the traditional functional model $\mathbf{y} = \mathbf{AX} + \mathbf{v}$ is split into the models $\mathbf{y} = \mathbf{AX}_{(1)} + \mathbf{v}_{(1)}$ and $\mathbf{y} = \mathbf{AX}_{(2)} + \mathbf{v}_{(2)}$, both of which are related to the same observation vector $\mathbf{y}$. The parameter vectors of such models are competitive to each other. It is obvious that the outlying observation can have more than one cluster. Then one can assume the split models in the form $\mathbf{y} = \mathbf{AX}_{(l)} + \mathbf{v}_{(l)}$, $l = 1, \ldots, q$.

Generally M_{split} estimation was design for arbitrary q ($M_{split(q)}$ estimation). However, this paper focuses on $M_{split(2)}$ estimation, where $M_{split(2)}$ estimates of the competitive parameters, namely $\hat{\mathbf{X}}_{(1)}$ and $\hat{\mathbf{X}}_{(2)}$, satisfy the equation

$$\min_{\mathcal{X}_2} \varphi(\mathcal{X}_2) = \min_{\mathcal{X}_2} \sum_{i=1}^{n} v_{i(1)}^2 v_{i(2)}^2 = \sum_{i=1}^{n} \hat{v}_{i(1)}^2 \hat{v}_{i(2)}^2, \tag{3}$$

where $\mathcal{X}_2 = (\mathbf{X}_{(1)}, \mathbf{X}_{(2)})$, $\hat{\mathbf{v}}_{(1)} = \mathbf{y} - \mathbf{AX}_{(1)}$ and $\hat{\mathbf{v}}_{(2)} = \mathbf{y} - \mathbf{AX}_{(2)}$. In teh case of arbitrary q one can write $\varphi(\mathcal{X}_q) = \sum_{i=1}^{n} \prod_{l=1}^{q} v_{i(l)}^2$. Note that, if $\forall l: \mathbf{X}_{(l)} = \mathbf{X} \iff v_{i(l)} = v_i$, then $\varphi(\mathcal{X}_q) = \varphi(\mathbf{X}) = \sum_{i=1}^{n} v_i^{2q}$.

Considering the objective function of the optimization problem Eq. (3), we can determine the following influence functions (the influence functions for both competitive parameters)

$$\psi_{(1)}(\boldsymbol{\mathcal{V}}_2) = \frac{\partial v_{(1)}^2 v_{(2)}^2}{\partial v_{(1)}} = 2v_{(1)}v_{(2)}^2; \quad \psi_{(2)}(\boldsymbol{\mathcal{V}}_2) = \frac{\partial v_{(1)}^2 v_{(2)}^2}{\partial v_{(1)}} = 2v_{(2)}v_{(1)}^2; \quad \boldsymbol{\mathcal{V}}_2 = (v_{(1)}, v_{(2)}). \tag{4}$$

Hence the weight functions have the respective forms: $w_{(1)}(\boldsymbol{\mathcal{V}}_2) = v_{(2)}^2$ or $w_{(2)}(\boldsymbol{\mathcal{V}}_2) = v_{(1)}^2$. $M_{split(2)}$ estimates, which solve the optimization problem of Eq. (3), also zero the influence functions of Eq. (4), which will be used in determination of the estimates in question.

Sensitivity of $M_{split(2)}$ estimates are considered within the interval $\Delta\theta = \theta_{(2)} - \theta_{(1)}$ for which we can write $\gamma_{(1)+}^* = \sup \psi_{(1)} > 0$, $\gamma_{(1)-}^* = \inf \psi_{(1)} = 0$, $\gamma_{(2)+}^* = \sup \psi_{(2)} = 0$, $\gamma_{(2)-}^* = \inf \psi_{(2)} < 0$. Thus, we obtain the following maximum sensitivities

$$\gamma_{(1)}^* = \max\{\gamma_{(1)+}^*, -\gamma_{(1)-}^*\} = \gamma_{(1)+}^*; \quad \gamma_{(2)}^* = \max\{\gamma_{(2)+}^*, -\gamma_{(2)-}^*\} = -\gamma_{(2)-}^*. \tag{5}$$

The sensitivities $\gamma_{(1)}^*$ and $\gamma_{(2)}^*$ are finite, hence $M_{split(2)}$ estimation is robust within the interval $\Delta\theta$. However, it is not robust outside that interval, thus it can break down if there are more than two observation clusters (in such a

case one should assume a different and appropriate number q). Note that if $\theta_{(1)} = \theta_{(2)} = \theta \iff v_{(1)} = v_{(2)} = v$, $\boldsymbol{\mathcal{V}}_2 = (v_{(1)}, v_{(2)}) = v$, then $\psi_{(1)}(\boldsymbol{\mathcal{V}}_2) = \psi_{(2)}(\boldsymbol{\mathcal{V}}_2) = \psi(v) = 2v^3$ $w_{(1)}(\boldsymbol{\mathcal{V}}_2) = w_{(2)}(\boldsymbol{\mathcal{V}}_2) = w(v) = 2v^2$, which are the influence and weight functions of LFP-method, respectively (Cellmer 2014). Since $\gamma_+^* = +\infty$, $\gamma_-^* = -\infty$, $\gamma^* = \max\{\gamma_+^*, -\gamma_-^*\} = \infty$, and the weight function is convex, thus LFP method is a weak estimation (like all other methods with the objective function $\varphi(\mathbf{X}) = \sum_{i=1}^n v_i^{2q}$).

M$_\mathcal{P}$ estimation and its robustness

M-estimates, which minimize the objective function of Eq. (1), are solutions of the following equation

$$\sum_{i=1}^n \psi(v_i) \frac{\partial v_i}{\partial \mathbf{X}} = \sum_{i=1}^n \psi(v_i) \mathbf{a}_i^T = \mathbf{0} \ . \tag{6}$$

Assuming that $\rho(v) = -\ln f(v)$, where $f = F'$ is a probability density function (PDF), M-estimates are determined for the certain family of distributions $F(y;\theta)$. In such a case $\psi(v) = \rho'(v) = -f'(v)/f(v)$. The problem is to assume a family of distributions, which is general enough on one hand, but yields no numerical problems on the other. Generality of the model is related especially to anomalies of empirical distributions, which usually concern asymmetry of a distribution $\beta_1 = \mu_3 / \mu_2^{3/2}$ and/or its kurtosis $\beta_2 = \mu_4 / \mu_2^2$; where μ_k is kth central moment (note that for normal distributions $\beta_1 = 0$, $\beta_2 = 3$). Such anomalies in geodetic measurements were pointed out in several papers (e. g., Wiśniewski 1996; Hu *et al.* 2001). If we want to consider such anomalies during the estimation process we should assume appropriate probabilistic models. In that context, the family of Pearson's distributions seems especially interesting. Fortunately, PDFs of Pearson's distributions are solutions of the following differential equation (e.g., Elderton 1953; Wiśniewski 2014)

$$-\frac{f'(v)}{f(v)} = \frac{(c_0 + 3c_2)v}{\sigma^2 c_0 - \sigma c_1(v-s) + c_2(v-s)^2} = \psi(v) = \psi(v; \beta_1, \beta_2), \tag{7}$$

where : $c_0 = 4\beta_2 - 3\beta_1^2$, $c_1 = \beta_1(\beta_2 + 3)$, $c_2 = 2\beta_2 - 3\beta_1^2 - 6$ (s – a difference between the mode and the expected value). Note that such an equation defines also the influence function (with the opposite sign). M-estimation with such an influence function was determined by Wiśniewski (2014) and it was denoted as M$_\mathcal{P}$ estimation.

The maximum sensitivity of M$_\mathcal{P}$ estimation can be described by the following expression

$$\gamma^* = \max\{\gamma_+^*, -\gamma_-^*\} = \max\{-\inf \psi(v; \beta_1, \beta_2), \sup \psi(v; \beta_1, \beta_2)\}. \tag{8}$$

If $\beta_2 > 3$ and $c_2 > 0$, then $\gamma^* < \infty$, and additionally the weight function $w = \psi/v$ is within the whole interval $(-\infty, \infty)$ a concave function with the maximum $\max w(v) = w(0)$. Thus, M$_\mathcal{P}$ estimation is robust, and its robustness increases with growing distance between the kurtosis and the value of $3/2\beta_1^2 + 3$ (in the case of symmetric distributions with the growing kurtosis). If $\beta_2 < 3$, then the weight function is convex, hence M$_\mathcal{P}$ estimation itself is a weak estimation. For $\beta_1 = 0$ and $\beta_2 = 3$, M$_\mathcal{P}$ estimation becomes LS method.

Example of numerical analysis

Qualitative robustness

The basis for the empirical analysis is an observation set with the elementary functional model $v_i = y_i - \theta$, $i = 1, \ldots, n$, which is simulated by applying the Gaussian generator of the system MatLab. Wiśniewski (2014) showed that in the case of small samples, the empirical kurtosis and asymmetry might differ from the theoretical values for the normal distribution. For example, an observation set was generated under the assumption that $n = 32$ and $\theta = 0$, and one obtained the following empirical values: the standard deviation $\sigma = 1.10$, the asymmetry $\beta_1 = 1.00$ and the kurtosis $\beta_2 = 5.60$. One of the simulated observations was disturbed with the growing gross error. The results obtained by applying LS method, M-estimation with the Huber influence function $\psi(v) = v \min\{1, k/|v|\}$ (for $k = 2.5$),

M$_{\mathcal{P}}$ or M$_{split(2)}$ estimation are presented in Table 1. Fig. 1 shows the histograms of the observation set for $g = 0$, $g = 10$, $g = 20$ or $g = 40$.

Table 1. Comparison M$_{\mathcal{P}}$ and M$_{split(2)}$ estimates with *LS* and Huber's estimates

	g	0	5	10	20	40	80	160	320
	$\hat{\theta}_{LS}$	0.28	0.43	0.59	0.90	1.52	2.77	5.27	10.27
	$\hat{\theta}_H$	0.21	0.31	0.27	0.27	0.27	0.35	0.46	0.46
	$\hat{\theta}_P$	−0.08	−0.02	−0.03	−0.04	−0.04	−0.05	−0.05	−0.05
M$_{split(2)}$	$\hat{\theta}_{(1)}$	−0.41	−0.30	0.10	0.17	0.24	0.27	0.29	0.30
	$\hat{\theta}_{(2)} = \hat{g}$	2.07	2.75	6.93	17.63	38.40	78.86	159.09	319.21

For such an asymmetric set of observations we can draw the following conclusions. First of all, if $g = 0$, then *LS* estimate and M estimate are close to each other ($\hat{\theta}_{LS} = 0.28$, $\hat{\theta}_H = 0.21$). M$_{\mathcal{P}}$ estimate, $\hat{\theta}_P = -0.08$, is close to the theoretical value of $\theta = 0$ (due to the fact that such an estimation considers the asymmetry of the set). Finally, M$_{split(2)}$ estiamtion reacted to positive asymmetry in the value of $\hat{\theta}_{(1)} = -0.41$ which is on the left hand side from the mode. The second value $\hat{\theta}_{(2)} = 2.07$ results from the fact that the method "regarded" the right hand side tail as the outlying observations. The situation changes with the growing value of the gross error. Then, M$_{split(2)}$ estimates, namely $\hat{\theta}_{(1)}$ and $\hat{\theta}_{(2)}$, identify two clusters of the observations in a better and better way.
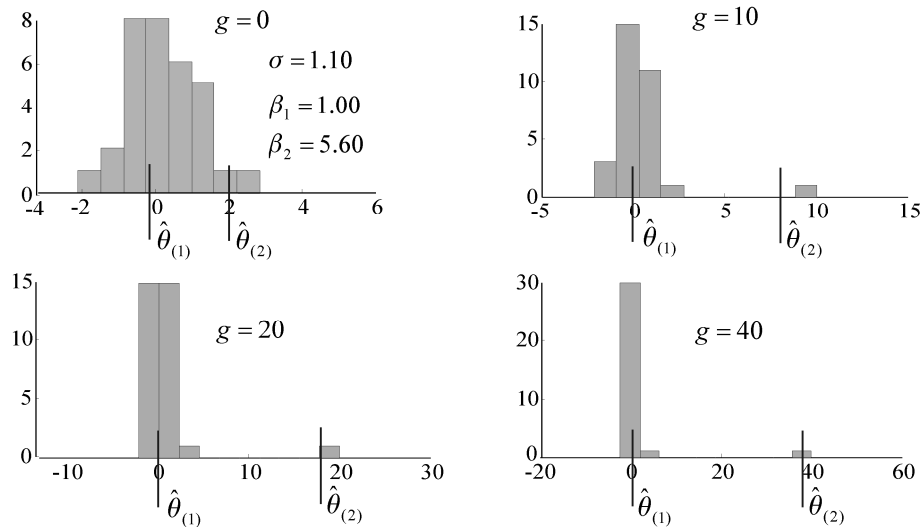

Fig. 1. Selected histograms for: $g = 0$, $g = 10$, $g = 20$ and $g = 40$

Quantitative robustness

Let us apply the observation set from the previous example to illustrate such kind of robustness in the case of M$_{split(2)}$ or M$_{\mathcal{P}}$ estimation. Thus, let the set be disturbed by the growing number (from $k = 1$ to $k = 32$) of gross errors of the same value $g = 10$. The parameter $\theta = 0$ will be estimated by using M$_{split(2)}$ or M$_{\mathcal{P}}$ estimation (assuming that $\beta_1 = 1.00$). For the sake of comparison, the parameter will also be estimated by applying LS or Huber's method. The results which are obtained for each of the methods are presented in Fig. 2. The breakdown points of M estimate as well as M$_{\mathcal{P}}$ estimates are also indicated in that figure. As for M$_{split(2)}$ estimation, Fig. 2 shows the point of "reversal", namely the point in which the estimates swap places with each other. The Huber estimate breaks down at $k = 15$ (the empirical breakdown point $\varepsilon^*_{\hat{\theta}_H} = 0.47$). The empirical breakdown point of $M_{split(2)}$ estimate is just the same,

$\varepsilon^*_{\text{split}(2)} = 0.47$; however, for $k = 16$ we can observe the reversal of the observation subsets, and the estimates $\hat{\theta}_{(1)}$, $\hat{\theta}_{(2)}$ swap places with each other. For $1 < k < 15$ the estimate $\hat{\theta}_{(1)}$ describes the parameter $\theta = 0$, while $\hat{\theta}_{(2)}$ identifies gross errors (the situation is the opposite for $16 \le k < 28$). Finally, M$_\mathcal{P}$ estimate can withstand the biggest number of outliers, hence it has also the biggest breakdown point, namely $\varepsilon^*_{\hat{\theta}_\mathcal{P}} = 0.59$.
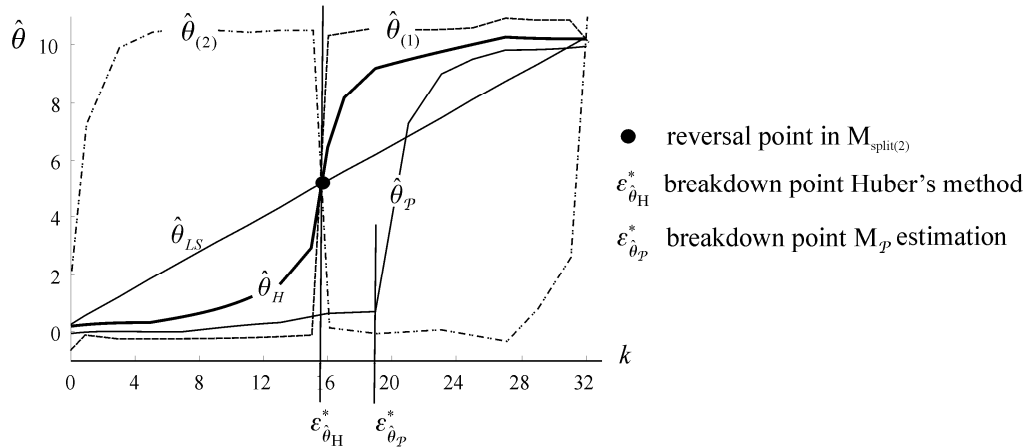

Fig. 2. Estimates $\hat{\theta}_{LS}$, $\hat{\theta}_H$, $\hat{\theta}_\mathcal{P}$ and M$_{\text{split}(2)}$ estimates $\hat{\theta}_{(1)}$, $\hat{\theta}_{(2)}$ depending on the growing number k of gross errors ($g = 10$)

Conclusions

The paper presents some interesting findings concerning robustness of the new methods of estimation, which can be applied in adjustment of geodetic measurements. However, if we consider M$_{\text{split}}$ or M$_\mathcal{P}$ estimation, the meaning of robustness is a little bit different from the traditional one. Generally speaking, the outlying observations are not regarded as "bad" ones. In M$_{\text{split}}$ estimation they are treated as realizations of a "strange" random variable (or variables) which parameters (like for example the expected value) differs from the parameters of "appropriate" variables. Hence, we should consider two (or more) functional models with the competitive parameter vectors. Estimating such two vectors, we can obtain information about the real parameters but also about competitive (strange) ones, which sometimes might be interesting and advisable (for example in deformation analyses). On the other hand, robustness of M$_\mathcal{P}$ estimation depends on the assumed (or estimated) asymmetry and kurtosis of error distribution. For certain types of distributions such a method is robust against outlying observations. If we assume proper values of the steering parameters then the method can accept a wider (or asymmetric) interval for measurement errors, hence some observations, which traditionally would be regarded as outlying ones, do not disturb the estimation results. It is worth noting that the breakdown point of M$_\mathcal{P}$ estimation can achieve values bigger than 0.5, thus it should be regarded as a subjective breakdown point (see, for example, Wyszkowska, Duchnowski 2017). Such breakdown points are of course related to the prior information about distribution of measurement errors, which in fact is the theoretical basis of M$_\mathcal{P}$ estimation. In summing up the numerical examples presented in this paper show that both methods in question are good and worth considering alternatives for traditional robust M-estimates.

Funding

References

Amiri-Simkooei, A. R.; Alaei-Tabatabaei, S. M.; Zangeneh-Nejad, F.; Voosoghi, B. 2016. Stability analysis of deformation-monitoring network points using simultaneous observation adjustment of two epochs, *Journal of Surveying Engineering* 143(1).

Błaszczak-Bąk, W.; Janowski, A.; Kamiński, W.; Rapiński, J. 2015. Application of the M$_{\text{split}}$ method for filtering airborne laser scanning data-sets to estimate digital terrain models, *International Journal of Remote Sensing* 36(9): 2421–2437. https://doi.org/10.1080/01431161.2015.1041617

Cellmer, S. 2014. Least fourth powers: optimisation method favouring outliers, *Survey Review* 47(345): 411–417. https://doi.org/10.1179/1752270614Y.0000000142

Duchnowski, R.; Wiśniewski, Z. 2011. Estimation of the shift between parameters of functional models of geodetic observations by applying M$_{\text{split}}$ estimation, *Journal of Surveying Engineering* 138(1): 1–8. https://doi.org/10.1061/(ASCE)SU.1943-5428.0000062

Duchnowski, R.; Wiśniewski, Z. 2014. Comparison of two unconventional methods of estimation applied to determine network point displacement, *Survey Review* 46(338): 401–405. https://doi.org/10.1179/1752270614Y.0000000127

Duchnowski, R.; Wiśniewski, Z. 2016. Accuracy of the Hodges–Lehmann estimates computed by applying Monte Carlo simulations, *Acta Geodaetica et Geophysica*: 1–15.

Elderton, W. P. 1953. *Frequency curves and correlation*. Cambridge University Press.

Ge, Y.; Yuan, Y.; Jia, N.; 2013. More efficient methods among commonly used robust estimation methods for GPS coordinate transformation, *Survey Review* 45: 229–234. https://doi.org/10.1179/1752270612Y.0000000028

Hampel, F. R.; Ronchetti, E. M.; Rousseuw, P. J.; Stahel, W. A. 1986. *Robust statistics. The approach based on influence functions*. John Wiley & Sons, New York.

Hu, X.; Huang, C.; Liao, X. 2001. A new solution assessment approach and its application to space geodesy data analysis, *Celestial Mechanics and Dynamical Astronomy* 81: 265–278. https://doi.org/10.1023/A:1013204418865

Huber, P. J. 1981. *Robust statistics*. John Wiley & Sons, New York. https://doi.org/10.1002/0471725250

Janicka, J.; Rapinski. J. 2013. M$_{\text{split}}$ transformation of coordinates, *Survey Review* 45: 269–274. https://doi.org/10.1179/003962613X13726661625708

Janowski, A.; Rapinski, J. 2013. M$_{\text{split}}$ estimation in laser scanning data modeling, *Journal of the Indian Society of Remote Sensing* 41(1): 15–19. https://doi.org/10.1007/s12524-012-0213-8

Kubik, K. 1982. An error theory for Danish method, in *Proceedings of Symposia Math Models, Accuracy Aspects and Quality Control*, ISP Comm. III, Helsinki.

Li, J.; Wang, A.; Xinyuan, W. 2013. M$_{\text{split}}$ estimate the relationship between LS and its application in gross error detection, *Mine Surveying (China)* 2: 57–59.

Wiśniewski, Z. 1996. Estimation of the third and fourth order central moments of measurement errors from sums of powers of least squares adjustment residuals, *Journal of Geodesy* 70: 256–262. https://doi.org/10.1007/s001900050016

Wiśniewski, Z. 2009. Estimation of parameters in a split functional model of geodetic observations (*M$_{\text{split}}$ estimation*), *Journal of Geodesy* 83: 105–120. https://doi.org/10.1007/s00190-008-0241-x

Wiśniewski, Z. 2010. *M$_{\text{split(q)}}$* estimation: estimation of parameters in a multi split functional model of geodetic observations, *Journal of Geodesy* 84: 355–372. https://doi.org/10.1007/s00190-010-0373-7

Wiśniewski, Z. 2014. M-estimation with probabilistic models of geodetic observations, *Journal of Geodesy* 88: 941–957. https://doi.org/10.1007/s00190-014-0735-7

Wiśniewski, Z.; Zienkiewicz, M. H. 2016. Shift- M$_{\text{split}}^{*}$ estimation in deformation analyses, *Journal of Surveying Engineering* 142(4): 1–13. https://doi.org/10.1061/(ASCE)SU.1943-5428.0000183

Wyszkowska, P.; Duchnowski R. 2017. Subjective breakdown points of R-estimators applied in deformation analysis, in *10th International Conference "Environmental Engineering"*, 27–28 April 2017, Vilnius, Lithuania.

Yang, Y.; He, H.; Xu, G. 2001. Adaptively robust filtering for kinematic geodetic positioning, *Journal of Geodesy* 75: 109–116. https://doi.org/10.1007/s001900000157

Zienkiewicz, M. H. 2015. Application of M$_{\text{split}}$ estimation to determine control points displacements in networks with unstable reference system, *Survey Review* 47(342): 174–180. https://doi.org/10.1179/1752270614Y.0000000105

Zienkiewicz, M. H.; Baryła, R. 2015. Determination of vertical indicators of ground deformation in the old and main city of Gdansk area by applying unconventional method of robust estimation, *Acta Geodynamica et Geomaterialia* 12, No. 3 (179): 249–257.